OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE UNIT **4727/01**

MATHEMATICS

Further Pure Mathematics 3

MONDAY 18 JUNE 2007 Morning

Time: 1 hour 30 minutes

Additional Materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

ADVICE TO CANDIDATES

- Read each question carefully and make sure you know what you have to do before starting your answer.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

© OCR 2007 [H/102/2700]

OCR is an exempt Charity

[Turn over

1 **(i)** By writing z in the form $re^{i\theta}$, show that $zz^* = |z|^2$. [1]

(ii) Given that $zz^* = 9$, describe the locus of z. [2]

2 A line l has equation $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$ and a plane Π has equation $8x - 7y + 10z = 7$. Determine whether l lies in Π, is parallel to Π without intersecting it, or intersects Π at one point. [5]

3 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{3x}.$$ [6]

4 Elements of the set $\{p, q, r, s, t\}$ are combined according to the operation table shown below.

	p	q	r	s	t
p	t	s	p	r	q
q	s	p	q	t	r
r	p	q	r	s	t
s	r	t	s	q	p
t	q	r	t	p	s

(i) Verify that $q(st) = (qs)t$. [2]

(ii) Assuming that the associative property holds for all elements, prove that the set $\{p, q, r, s, t\}$, with the operation table shown, forms a group G. [4]

(iii) A multiplicative group H is isomorphic to the group G. The identity element of H is e and another element is d. Write down the elements of H in terms of e and d. [2]

5 **(i)** Use de Moivre's theorem to prove that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$ [4]

(ii) Hence find the largest positive root of the equation

$$64x^6 - 96x^4 + 36x^2 - 3 = 0,$$

giving your answer in trigonometrical form. [4]

© OCR 2007

6 Lines l_1 and l_2 have equations

$$\frac{x-3}{2} = \frac{y-4}{-1} = \frac{z+1}{1} \qquad \text{and} \qquad \frac{x-5}{4} = \frac{y-1}{3} = \frac{z-1}{2}$$

respectively.

(i) Find the equation of the plane Π_1 which contains l_1 and is parallel to l_2, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. [5]

(ii) Find the equation of the plane Π_2 which contains l_2 and is parallel to l_1, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. [2]

(iii) Find the distance between the planes Π_1 and Π_2. [2]

(iv) State the relationship between the answer to part **(iii)** and the lines l_1 and l_2. [1]

7 **(i)** Show that $(z - e^{i\phi})(z - e^{-i\phi}) \equiv z^2 - (2\cos\phi)z + 1$. [1]

(ii) Write down the seven roots of the equation $z^7 = 1$ in the form $e^{i\theta}$ and show their positions in an Argand diagram. [4]

(iii) Hence express $z^7 - 1$ as the product of one real linear factor and three real quadratic factors. [5]

8 **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + y\tan x = \cos^3 x,$$

expressing y in terms of x in your answer. [8]

(ii) Find the particular solution for which $y = 2$ when $x = \pi$. [2]

9 The set S consists of the numbers 3^n, where $n \in \mathbb{Z}$. ($\mathbb{Z}$ denotes the set of integers $\{0, \pm1, \pm2, \ldots\}$.)

(i) Prove that the elements of S, under multiplication, form a commutative group G. (You may assume that **addition** of integers is associative and commutative.) [6]

(ii) Determine whether or not each of the following subsets of S, under multiplication, forms a subgroup of G, justifying your answers.

(a) The numbers 3^{2n}, where $n \in \mathbb{Z}$. [2]

(b) The numbers 3^n, where $n \in \mathbb{Z}$ and $n \geqslant 0$. [2]

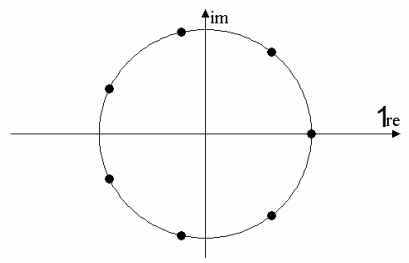
(c) The numbers $3^{(\pm n^2)}$, where $n \in \mathbb{Z}$. [2]

© OCR 2007

1 **(i)** $z z^* = r\,\mathrm{e}^{\mathrm{i}\theta}.r\,\mathrm{e}^{-\mathrm{i}\theta} = r^2 = \lvert z\rvert^2$	B1 **1**	For verifying result **AG**
(ii) Circle Centre $0\,(+0\mathrm{i})$ *OR* (0, 0) *OR O*, radius 3	B1 B1 **2** **3**	For stating circle For stating correct centre and radius
2 *EITHER*: $(\mathbf{r} =)\,[3+t, 1+4t, -2+2t]$ $8(3+t)-7(1+4t)+10(-2+2t) = 7$ $\Rightarrow (0t)+(-3) = 7 \Rightarrow$ contradiction l is parallel to Π, no intersection	M1 M1 A1 A1 B1 **5**	For parametric form of l seen or implied For substituting into plane equation For obtaining a contradiction For conclusion from correct working
OR: $[1, 4, 2]\,.\,[8, -7, 10] = 0$ $\Rightarrow l$ is parallel to Π $(3, 1, -2)$ into Π $\Rightarrow\ 24-7-20 \neq 7$ l is parallel to Π, no intersection	M1 A1 M1 A1 B1	For finding scalar product of direction vectors For correct conclusion For substituting point into plane equation For obtaining a contradiction For conclusion from correct working
OR:Solve $\dfrac{x-3}{1} = \dfrac{y-1}{4} = \dfrac{z+2}{2}$ and $8x-7y+10z = 7$ eg $y-2z = 3$, $2y-2 = 4z+8$ eg $4z+4 = 4z+8$ l is parallel to Π, no intersection	 M1 A1 M1 A1 B1 **5**	 For eliminating one variable For eliminating another variable For obtaining a contradiction For conclusion from correct working
3 Aux. equation $m^2-6m+8\,(=0)$ $m = 2, 4$ CF $(y =)\ A\mathrm{e}^{2x} + B\mathrm{e}^{4x}$ PI $(y =)\ C\mathrm{e}^{3x}$ $9C-18C+8C = 1 \Rightarrow C = -1$ GS $y = A\mathrm{e}^{2x} + B\mathrm{e}^{4x} - \mathrm{e}^{3x}$	M1 A1 A1√ M1 A1 B1√ **6** **6**	For auxiliary equation seen For correct roots For correct CF. f.t. from their m For stating and substituting PI of correct form For correct value of C For GS. f.t. from their CF + PI with 2 arbitrary constants in CF and none in PI

4 **(i)** $q(st) = qp = s$ $(qs)t = tt = s$	B1 B1 **2**	For obtaining s For obtaining s
(ii) METHOD 1 Closed: see table Identity = r Inverses: $p^{-1} = s,\ q^{-1} = t,\ (r^{-1} = r),$ $s^{-1} = p,\ t^{-1} = q$	 B1 B1 M1 A1 **4**	 For stating closure with reason For stating identity r For checking for inverses For stating inverses *OR* For giving sufficient explanation to justify each element has an inverse eg r occurs once in each row and/or column
METHOD 2 Identity = r eg $p^2 = t$, $p^3 = q$, $p^4 = s$ $\Rightarrow p^5 = r$, so p is a generator	 B1 M1 A1 A1	 For stating identity r For attempting to establish a generator $\neq r$ For showing powers of p (*OR* q, s or t) are different elements of the set For concluding p^5 (*OR* q^5, s^5 or t^5) $= r$
(iii) e, d, d^2, d^3, d^4	B2 **2**	For stating all elements **AEF** eg d^{-1}, d^{-2}, dd
	8	

5 **(i)** $(\cos 6\theta =)\ \mathrm{Re}(c + \mathrm{i}s)^6$	M1	For expanding (real part of) $(c + \mathrm{i}s)^6$ at least 4 terms and 1 evaluated binomial coefficient needed
$(\cos 6\theta =)\ c^6 - 15c^4s^2 + 15c^2s^4 - s^6$	A1	For correct expansion
$(\cos 6\theta =)$ $c^6 - 15c^4(1 - c^2) + 15c^2(1 - c^2)^2 - (1 - c^2)^3$	M1	For using $s^2 = 1 - c^2$
$(\cos 6\theta =)\ 32c^6 - 48c^4 + 18c^2 - 1$	A1 **4**	For correct result **AG**
(ii) $64x^6 - 96x^4 + 36x^2 - 3 = 0 \Rightarrow \cos 6\theta = \frac{1}{2}$	M1	For obtaining a numerical value of $\cos 6\theta$
$\Rightarrow (\theta =)\ \frac{1}{18}\pi, \frac{5}{18}\pi, \frac{7}{18}\pi$ etc.	A1	For any correct solution of $\cos 6\theta = \frac{1}{2}$
$\cos 6\theta = \frac{1}{2}$ has multiple roots	M1	For stating or implying at least 2 values of θ
largest x requires smallest θ $\Rightarrow$ largest positive root is $\cos \frac{1}{18}\pi$	A1 **4**	For identifying $\cos \frac{1}{18}\pi$ **AEF** as the largest positive root from a list of 3 positive roots *OR* from general solution *OR* from consideration of the cosine function
	8	

6 **(i)** $\mathbf{n} = l_1 \times l_2$	B1	For stating or implying in **(i)** or **(ii)** that **n** is perpendicular to l_1 and l_2
$\mathbf{n} = [2, -1, 1] \times [4, 3, 2]$	M1*	For finding vector product of direction vectors
$\mathbf{n} = k[-1, 0, 2]$	A1	For correct vector (any k)
$[3, 4, -1] \,.\, k[-1, 0, 2] = -5k$	M1 (*dep)	For substituting a point of l_1 into $\mathbf{r}.\mathbf{n}$
$\mathbf{r}.[-1, 0, 2] = -5$	A1 **5**	For obtaining correct p. **AEF** in this form
(ii) $[5, 1, 1] \,.\, k[-1, 0, 2] = -3k$	M1	For using same **n** and substituting a point of l_2
$\mathbf{r}.[-1, 0, 2] = -3$	A1√ **2**	For obtaining correct p. **AEF** in this form f.t. on incorrect **n**
(iii) $d = \dfrac{\lvert -5+3 \rvert}{\sqrt{5}}$ *OR* $d = \dfrac{\lvert [2,-3,2] \,.\, [-1,0,2] \rvert}{\sqrt{5}}$	M1	For using a distance formula from their equations Allow omission of | |
OR d from $(5, 1, 1)$ to $\Pi_1 = \dfrac{\lvert 5(-1)+1(0)+1(2)+5 \rvert}{\sqrt{5}}$		
OR d from $(3, 4, -1)$ to $\Pi_2 = \dfrac{\lvert 3(-1)+4(0)-1(2)+3 \rvert}{\sqrt{5}}$		
OR $[3-t, 4, -1+2t] \,.\, [-1, 0, 2] = -3 \Rightarrow t = \frac{2}{5}$		*OR* For finding intersection of $\mathbf{n}_1$ and Π_2 or $\mathbf{n}_2$ and Π_1
OR $[5-t, 1, 1+2t] \,.\, [-1, 0, 2] = -5 \Rightarrow t = -\frac{2}{5}$		
$d = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5} = 0.894427...$	A1√ **2**	For correct distance **AEF** f.t. on incorrect **n**
(iv) d is the shortest *OR* perpendicular distance between l_1 and l_2	B1 **1**	For correct statement
	10	
7 **(i)** $(z - e^{i\phi})(z - e^{-i\phi}) \equiv z^2 - (2)z\dfrac{(e^{i\phi} + e^{-i\phi})}{(2)} + 1$	B1 **1**	For correct justification **AG**
$\equiv z^2 - (2\cos\phi)z + 1$		
(ii) $z = e^{\frac{2}{7}k\pi i}$	B1	For general form *OR* any one non-real root
for $k = 0, 1, 2, 3, 4, 5, 6$ *OR* $0, \pm 1, \pm 2, \pm 3$	B1	For other roots specified (k=0 may be seen in any form, eg 1, e^0, $e^{2\pi i}$)
		For answers in form $\cos\theta + i\sin\theta$ allow maximum B1 B0
	B1	For any 7 points equally spaced round unit circle (circumference need not be shown)
	B1 **4**	For 1 point on $+^{ve}$ real axis, and other points in correct quadrants
(iii) $\left(z^7 - 1 = \right)(z-1)(z - e^{\frac{2}{7}\pi i})(z - e^{\frac{4}{7}\pi i})$	M1	For using linear factors from **(ii),** seen or implied
$(z - e^{\frac{6}{7}\pi i})(z - e^{\frac{-2}{7}\pi i})(z - e^{\frac{-4}{7}\pi i})(z - e^{\frac{-6}{7}\pi i})$		
$= (z - e^{\frac{2}{7}\pi i})(z - e^{\frac{-2}{7}\pi i}) \times (z - e^{\frac{4}{7}\pi i})(z - e^{\frac{-4}{7}\pi i})$	M1	For identifying at least one pair of complex conjugate factors
$(z - e^{\frac{6}{7}\pi i})(z - e^{\frac{-6}{7}\pi i}) \times$		
$\times (z-1)$	B1	For linear factor seen
$= (z^2 - (2\cos\frac{2}{7}\pi)z + 1) \times$	A1	For any one quadratic factor seen
$(z^2 - (2\cos\frac{4}{7}\pi)z + 1) \times (z^2 - (2\cos\frac{6}{7}\pi)z + 1) \times$	A1 **5**	For the other 2 quadratic factors and expression written as product of 4 factors
$\times (z-1)$		
	10	

8 **(i)** Integrating factor $e^{\int \tan x\,(dx)}$	B1	For correct IF
$= e^{-\ln \cos x}$	M1	For integrating to ln form
$= (\cos x)^{-1}$ *OR* $\sec x$	A1	For correct simplified IF **AEF**
$\Rightarrow \frac{d}{dx}\left(y(\cos x)^{-1}\right) = \cos^2 x$	B1√	For $\frac{d}{dx}(y.\text{their IF}) = \cos^3 x$. their IF
	M1	For integrating LHS
$y(\cos x)^{-1} = \int \frac{1}{2}(1+\cos 2x)\,(dx)$	M1	For attempting to use $\cos 2x$ formula *OR* parts for $\int \cos^2 x\,dx$
$y(\cos x)^{-1} = \frac{1}{2}x + \frac{1}{4}\sin 2x\,(+c)$	A1	For correct integration both sides **AEF**
$y = \left(\frac{1}{2}x + \frac{1}{4}\sin 2x + c\right)\cos x$	A1 **8**	For correct general solution **AEF**
(ii) $2 = \left(\frac{1}{2}\pi + c\right).-1 \Rightarrow c = -2 - \frac{1}{2}\pi$	M1	For substituting $(\pi, 2)$ into their GS and solve for c
$y = \left(\frac{1}{2}x + \frac{1}{4}\sin 2x - 2 - \frac{1}{2}\pi\right)\cos x$	A1 **2**	For correct solution **AEF**
	10	
9 **(i)** $3^n \times 3^m = 3^{n+m}$, $n+m \in \mathbb{Z}$	B1	For showing closure
$\left(3^p \times 3^q\right)\times 3^r = \left(3^{p+q}\right)\times 3^r = 3^{p+q+r}$	M1	For considering 3 distinct elements, seen bracketed 2+1 or 1+2
$= 3^p \times \left(3^{q+r}\right) = 3^p \times \left(3^q \times 3^r\right) \Rightarrow$ associativity	A1	For correct justification of associativity
Identity is 3^0	B1	For stating identity. Allow 1
Inverse is 3^{-n}	B1	For stating inverse
$3^n \times 3^m = 3^{n+m} = 3^{m+n} = 3^m \times 3^n \Rightarrow$ commutativity	B1 **6**	For showing commutativity
(ii) (a) $3^{2n} \times 3^{2m} = 3^{2n+2m}\left(= 3^{2(n+m)}\right)$	B1*	For showing closure
Identity, inverse OK	B1 (*dep) **2**	For stating other two properties satisfied and hence a subgroup
(b) For 3^{-n},	M1	For considering inverse
$-n \notin$ subset	A1 **2**	For justification of not being a subgroup 3^{-n} must be seen here or in **(i)**
(c) *EITHER*: eg $3^{1^2} \times 3^{2^2} = 3^5$	M1	For attempting to find a specific counter-example of closure
$\neq 3^{r^2} \Rightarrow$ not a subgroup	A1 **2**	For a correct counter-example and statement that it is not a subgroup
OR: $3^{n^2} \times 3^{m^2} = 3^{n^2+m^2}$	M1	For considering closure in general
$\neq 3^{r^2}$ eg $1^2 + 2^2 = 5 \Rightarrow$ not a subgroup	A1	For explaining why $n^2 + m^2 \neq r^2$ in general and statement that it is not a subgroup
	12	